Squaring the Circle - statistically speaking

KEYWORDS:
Teaching;
Circle, needle and rectangle problems.

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Summary
A simple relationship is described which connects Buffon's classical needle problem with related problems involving circles, squares and rectangles.

◆INTRODUCTION ◆

A SIMPLE game to play with children involves tossing or rolling circular hoopla rings on to a floor comprising boards of equal width. The aim is to make the ring come to rest *between* the cracks, i.e. not to cross them. Following a brief derivation of the associated probability of such an event occurring (assuming a random toss or roll), we note a relationship that connects this problem to Buffon's well-known needle problem. This leads on to further investigation which we initiate, and then leave readers to follow this up.

◆RING PROBLEM◆

Suppose a ring of diameter *d* is dropped or rolled on to a floor comprising wooden planks (floorboards) of width *b*. The aim is to determine the probability, *p*, that the ring crosses a crack between the planks. Figure 1 shows one such plank with the ring on it

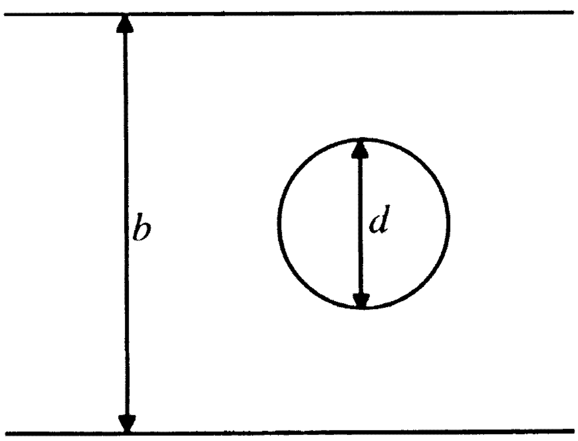

Fig 1.

Clearly, if $d > b$ then the ring will always cross a crack, so that the probability $p = 1$. (This assumes that the unlikely event of the ring landing on its edge does not occur.) If $d \le b$, however, then the probability

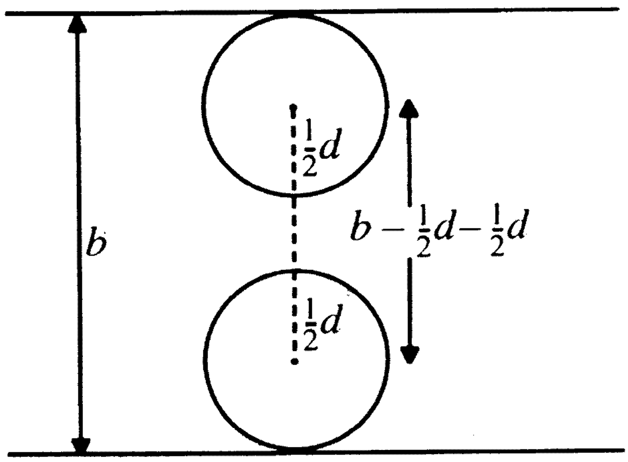

Fig 2.

that the ring crosses a crack is given by

$$p = 1 - q$$

where q = probability that the ring does *not* cross a crack. (Note that if a ring touches a crack then it is deemed to have crossed it.)

Referring to figure 2, the probability q is given by

q = (length over which the centre of the ring can land between upper and lower cracks without crossing either crack)/ (length over which the centre of the ring can land between upper and lower cracks)

$$= \frac{b - \frac{1}{2}d - \frac{1}{2}d}{b} = \frac{b - d}{b}$$

by considering the limiting cases of touching upper and lower cracks, and noting that the orientation of the ring does not matter. Thus the probability that the ring *will* cross a crack is

$p = 1 - q = 1 - (b - d)/b = d/b$ for $d \le b$

If we rewrite this probability in terms of the perimeter or circumference of the ring, which is πd, then

$p = (\text{perimeter})/\pi b$ (1)

provided the *perimeter* $= \pi d \le \pi b$.

◆COMPARISON WITH ◆ NEEDLE PROBLEM

Having determined this simple probability, I became interested in comparing it with the result for Buffon's classical needle problem. If a needle of length l is dropped on to floorboards of width b, with $l \leq b$, then the probability that the needle crosses a crack is given by

$$p = 2l/\pi b \qquad (2)$$

One interpretation of the expression in (2) is as follows. The needle can be thought of as a rectangle of breadth l and zero width, and with this interpretation the needle has perimeter $2l + 2.0 = 2l$. Hence (2) can be rewritten as

p = perimeter/πb

provided the *perimeter = 21* $\leq 2b$, and is therefore the same result as for the ring (1). Thus the formula (1) holds for both the ring and the needle provided the *perimeter* $\leq \min(2b, \pi b) = 2b$.

The obvious question to ask is whether this connection is also true for rectangles of non-zero width.

◆RECTANGLE PROBLEM◆

Consider now the case where the thrown object is in the shape of a rectangle of dimensions $c \times d$. This will include, for example, a playing card, as well as a piece of wire bent into the shape of a rectangle. For simplicity we consider the case where the largest dimension of the card is less than the width of the boards, i.e. the diagonal $\sqrt{(c^2 + d^2)} \leq b$. If this is not the case then the problem becomes much harder.

Figure 3 shows one such plank with the rectangle on it. The probability that the rectangle crosses a

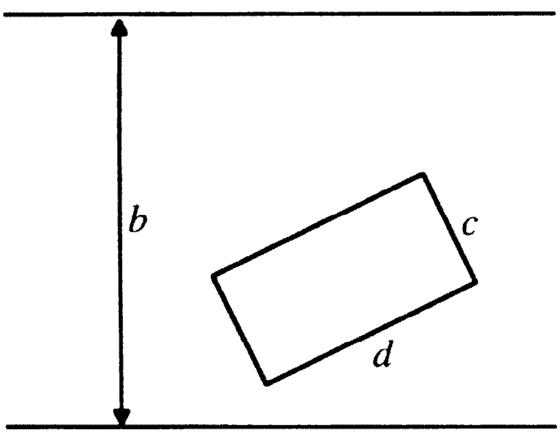

Fig 3.

crack can be determined, as before, from $p = 1 - q$ where q = probability that the rectangle does *not* cross a crack. To determine q we consider the length which the centre of the rectangle can land so that it does not cross a crack, and figure 4 shows the required length as a dotted line. In contrast to the ring problem, however we see that this time this length depends on the *orientation* of the rectangle. It is this feature that makes this problem harder than the corresponding ring problem, and similar to the well-known needle problem

To assign a description of a particular orientation we measure the anti-clockwise angle between *CD* and

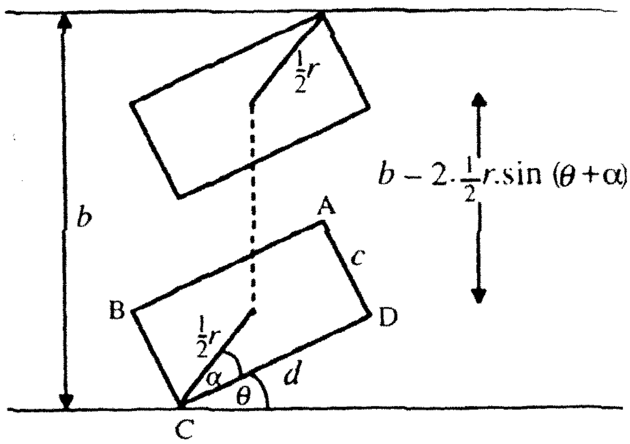

Fig 4.

"horizontal". Denoting this angle by θ, we see that, for an acute angle $0 \leq \theta \leq \pi/2$, the length of the dotted line in figure 4 is

$$b - 2.(1/2)r\sin(\theta + \alpha) = b - r\sin(\theta + \alpha)$$

where $r = \sqrt{(c^2 + d^2)}$ is the diagonal of the rectangle and $\tan \alpha = c/d$ This length can be simplified further as

$b - r \sin\theta \cos\alpha - r \sin\alpha \cos\theta = b - d \sin\theta - c \cos\theta$

as $\sin\alpha = c/r$ and $\cos\alpha = d/r$ The total length over which the centre of the rectangle can land is b.

Now, since there is a different length for *each* orientation, we must calculate q from

q = (sum of outcomes for which the rectangle does not cross a crack)/(sum of possible outcomes)

where the 'sum' in both the numerator and denominator is obtained by integrating (with respect to θ) the corresponding lengths over which the rectangle can land, i.e.

q = (integral of length over which the centre of the rectangle can land so that the rectangle does not cross a crack)/(integral of possible length over which the centre of the rectangle can land)

(Note that θ has a uniform distribution.) To calculate the integrals we could either consider θ to range from 0 to 2π, and determine the corresponding length to that above in the case when $\pi/2 \le \theta \le \pi$ etc., or use symmetry to only integrate over θ in the range $0 \le \theta \le \pi/2$, and then multiply the answer by 4. However, the 4 will appear on both the numerator and denominator, and thus

$$q = \frac{\int_0^{\frac{\pi}{2}} (b - d\sin\theta - c\sin\theta)d\theta}{\int_0^{\frac{\pi}{2}} bd\theta}$$

$$= \frac{[b\theta = d\cos\theta - c\sin\theta]_0^{\frac{\pi}{2}}}{[b\theta]_0^{\frac{\pi}{2}}}$$

$$= \frac{\frac{1}{2}\pi b - c - d}{\frac{1}{2}\pi b} = 1 - \frac{2(c+d)}{\pi b}$$

The probability that the rectangle does cross a crack is then

$p = 1- q = 2(c + d)/\pi b$ (3)

In terms of the perimeter of the rectangle, 2 *(c + d)*, the probability in (3) can be rewritten as
p = perimeter/*πb* (4)
where $\sqrt{(c^2 + d^2)} \le b$. Note that this result also includes the special case of a square where *c = d*, with the condition that $\sqrt{2}c \le b$, i.e perimeter = 4c $\le 2\sqrt{2}b$

◆COMPARISON WITH◆ RING PROBLEM

From (1) and (4) we see that, if a circular ring and a rectangular wire have the same perimeter, then the probability that they cross a crack when dropped on to floorboards of width *b* is the same, and is given by *perimeter/πb* . (Note that the perimeter is restricted so that it is possible for the ring and rectangle to fit between the cracks for any orientation.) Equivalently, for a given distance between the floorboards, the probability depends only on the perimeter of the shape. This is true for circular rings and all rectangles of the same perimeter, including those of zero width and forming a needle, subject to a restriction on the ratio *perimeter/b.*

For example, the floorboards in a typical house are 140 mm wide, and the probability that a square of side 70mm will cross a crack when dropped is 4x70/140π = 2/π = 0.637 . A rectangle of dimensions 105 mm x 35 mm with aspect ratio 3:1 has the same perimeter as the square, and therefore the same probability of crossing a crack. Similarly, a circular ring of diameter 280/π = 89.1 mm has the same probability of crossing a crack, as does a needle of length 140 mm.

◆FURTHER WORK◆

We suggest experimental investigation of the results given here, and then further investigation to ascertain whether this result extends to other shapes. If the experimental evidence suggests that this is the case, then a formal proof can be sought.