

A Probability Game

Dear Sir

I read with interest "A Probability Game" by Jean Melrose in the Summer 1998 issue of *Teaching Statistics*, and have some comments.

The possible positions of a player in the simplified Snakes and Ladders game may be viewed as the states of a stationary Markov Chain. Consequently, "finding the probability of taking exactly n moves", $n > 2$, to finish the game, which Melrose rightly claims as "more difficult" than is the case $n = 2$, is still easily tackled by students who have had just a bit of matrix algebra. In particular, if P is the one-step probability transition matrix, then the appropriate element of P^n (see Kemeny and Snell (1960), for instance, for details) will give the chance of finishing in **at most** n moves. The chance of finishing in exactly n moves then, is an element of P^n minus an element of P^{n-1} . Kemeny and Snell (1960) also explain how to compute the expected game length from P (relatively easy) and the standard deviation of game length (more difficult).

The interested reader may wish to consult Althoen, King and Schilling (1993) or Gadbois (1993) for further details concerning the full version of Snakes and Ladders to share with his or her students. The results in these articles are readily adapted for the simplified games presented by Melrose (1998).

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P.S. Some details: For the game of figure 1 of Melrose (1998) with a fair die and the stated rules, the one-step probability transition matrix P is

$$P = \begin{array}{c|ccccc} & 0 & 1 & 3 & 4 & 6 \\ \hline 0 & 0 & \frac{1}{6} & \frac{2}{6} & \frac{2}{6} & \frac{1}{6} \\ 1 & 0 & 0 & \frac{2}{6} & \frac{2}{6} & \frac{2}{6} \\ 3 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & \frac{4}{6} \\ 4 & 0 & 0 & \frac{1}{6} & 0 & \frac{5}{6} \\ 6 & 0 & 0 & 0 & 0 & 1 \end{array}$$

The possible positions or states are 0, 1, 3, 4, 6 with 0 corresponding to the starting position off the board. The value $4/6$, for example, is the chance of going from 3 to 6 in exactly one turn. Note that players do

not remain in states 2 or 5 at the completion of a turn so that these values are excluded from the list of states.

Consequently,

$$P^2 = \begin{bmatrix} 0 & 0 & \frac{6}{36} & \frac{4}{36} & \frac{26}{36} \\ 0 & 0 & \frac{4}{36} & \frac{2}{36} & \frac{30}{36} \\ 0 & 0 & \frac{2}{36} & \frac{1}{36} & \frac{33}{36} \\ 0 & 0 & \frac{1}{36} & \frac{1}{36} & \frac{34}{36} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} P(\text{finishing in exactly 2 moves}) &= P(\text{finishing in at most 2 moves}) - P(\text{finishing in at most 1 move}) \\ &= (P^2)_{0,6} - (P^1)_{0,6} \\ &= \frac{26}{36} - \frac{1}{6} \\ &= \frac{5}{9} \end{aligned}$$

which agrees with Melrose (1998) but can be generalised to

$$\begin{aligned} P(\text{finishing in exactly } n \text{ moves}) &= P(\text{finishing in at most } n \text{ moves}) - P(\text{finishing in at most } n-1 \text{ moves}) \\ &= (P^n)_{0,6} - (P^{n-1})_{0,6} \end{aligned}$$

(Key fact: $(P^n)_{i,j}$ is the chance that you will end up in state j in n turns starting from state i .)

References

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- Melrose, J. (1998). A Probability Game. *Teaching Statistics*, 20(2), 53-54.